

# $K$ -THEORETIC POIRIER-REUTENAUER BIALGEBRA

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**ABSTRACT.** We use the  $K$ -Knuth equivalence of Buch and Samuel [3] to define a  $K$ -theoretic analogue of the Poirier-Reutenauer Hopf algebra. As an application, we rederive the  $K$ -theoretic Littlewood-Richardson rules of Thomas and Yong [18, 19] and of Buch and Samuel [3].

## 1. INTRODUCTION

**1.1. Poirier-Reutenauer Hopf algebra.** In [13], Poirier and Reutenauer defined a Hopf algebra structure on the  $\mathbb{Z}$ -span of all standard Young tableaux, which was later studied, for example, in [14] and [4]. Implicitly, this algebra also appears in [10]. Let us briefly recall the definition and illustrate it with few examples.

A Young diagram or partition is a finite collection of boxes arranged in left-justified rows such that the lengths of the rows are weakly decreasing from top to bottom. We denote the shape of a Young diagram  $\lambda$  by  $(\lambda_1, \lambda_2, \dots, \lambda_k)$ , listing the lengths of each row,  $\lambda_i$ . A Young tableau is a filling of the boxes of a Young diagram with positive integers so that the fillings increase in rows and columns. We call a Young tableau a standard Young tableau if it is filled with positive integers  $[k]$  for some  $k$ , where each integer appears exactly once. The tableau shown below is an example of a standard Young tableau of shape  $(3, 3, 2)$ .

1	4	6
2	5	7
3	8	

Given two partitions,  $\lambda$  and  $\mu$ , such that  $\mu \subset \lambda$ , we define the skew diagram  $\lambda/\mu$  to be the set of boxes of  $\lambda$  that do not belong to  $\mu$ . If the shape of  $T$  is  $\lambda/\mu$  where  $\mu$  is the empty shape, we say that  $T$  is of straight shape. The definitions of Young tableaux and standard Young tableaux extend naturally to skew diagrams. For example, the figure below shows a standard Young tableau of skew shape  $(3, 3, 1)/(2, 1)$ .

		3
	1	4
2		

Given a possibly skew Young tableau  $T$ , its row reading word,  $\mathbf{row}(T)$ , is obtained by reading the entries in the rows of  $T$  from left to right starting with the bottom row and ending with the top row. For the first standard Young tableau shown above,  $\mathbf{row}(T) = 38257146$ , and for the standard Young tableau of skew shape,  $\mathbf{row}(T) = 2143$ .

Next, consider words with distinct letters on some ordered alphabet  $A$ . We have the following *Knuth relations*:

$$pqs \approx qps \text{ and } sqp \approx spq \text{ whenever } p < s < q.$$

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Given two words,  $w_1$  and  $w_2$ , we say that they are Knuth equivalent, denoted  $w_1 \approx w_2$ , if  $w_2$  can be obtained from  $w_1$  by a finite sequence of Knuth relations. For example,  $52143 \approx 25143$  because

$$52143 \approx 52413 \approx 25413 \approx 25143.$$

If  $T_1$  and  $T_2$  are two tableaux, we say that  $T_1 \approx T_2$  if  $\mathbf{row}(T_1) \approx \mathbf{row}(T_2)$ . For example,

$$T_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \approx T_2 = \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 3 & \\ \hline 1 & & \\ \hline \end{array}.$$

From Theorem 5.2.5 of [10], any word with letters exactly  $[k]$  is Knuth equivalent to  $\mathbf{row}(T)$  for a unique standard Young tableau  $T$  of straight shape. This unique standard Young tableau may be obtained via RSK insertion of the word (see [16]). For example,  $52143 \approx \mathbf{row}(T)$  for

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}.$$

For a standard Young tableau  $T$ , let  $\mathbf{T} = \sum_{w \approx \mathbf{row}(T)} w$ . In other words,  $\mathbf{T}$  is the sum of words that are Knuth equivalent to  $\mathbf{row}(T)$ . Let  $PR$  be the  $\mathbb{R}$ -vector space generated by the set of  $\mathbf{T}$  for all standard Young tableaux.

Following [13], we next describe a bialgebra structure on  $PR$ . Start with two words,  $w_1$  and  $w_2$ , in  $PR$ , where  $w_1$  has letters exactly  $[n]$  for some positive integer  $n$ . Define  $w_2[n]$  to be the word obtained by adding  $n$  to each letter of  $w_2$ . Now define the product  $w_1 * w_2$  to be  $w_1 \sqcup w_2[n]$ , the shuffle product of  $w_1$  and  $w_2[n]$ . For example,  $12 * 1 = 12 \sqcup 3 = 123 + 132 + 312$ .

For a word  $w$  without repeated letters, define  $st(w)$  to be the unique word on  $\{1, 2, \dots, |w|\}$  obtained by applying the unique order-preserving injective mapping from the letters of  $w$  onto  $\{1, 2, \dots, |w|\}$  to the letters of  $w$ . For example,  $st(1426) = 1324$ . Then define the coproduct on  $PR$  by defining

$$\Delta(w) = \sum st(u) \otimes st(v),$$

where the sum is over all words  $u$  and  $v$  such that  $w$  is the concatenation of  $u$  and  $v$ . For example,  $\Delta(312) = \emptyset \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \emptyset$ , where  $\emptyset$  denotes the empty word. As shown in [13], the vector space  $PR$ , where we extend product  $*$  and coproduct  $\Delta$  by linearity, forms a bialgebra.

**1.2. Two versions of the Littlewood-Richardson rule.** While being interesting in its own right, the Poirier-Reutenauer Hopf algebra allows us to obtain a version of the Littlewood-Richardson rule for the cohomology rings of Grassmannians. In other words, it yields an explicitly positive description for the structure constants of the cohomology ring in the basis of *Schubert classes*. It is well-known that the Schubert classes can be represented by Schur functions of partitions that fit inside a rectangle. Thus, an essentially equivalent formulation of the problem is to describe structure constants of the ring of symmetric functions in terms of the basis of Schur functions. We refer the reader to [11] for a great introduction to the subject.

To see how the Poirier-Reutenauer Hopf algebra helps, let us state the following theorems.

**Theorem 1.1.** [10, Theorem 5.4.3] *Let  $T_1$  and  $T_2$  be two standard Young tableaux. Then we have*

$$\mathbf{T}_1 * \mathbf{T}_2 = \sum_{T \in T(T_1 \sqcup T_2)} \mathbf{T},$$

where  $T(T_1 \sqcup T_2)$  is the set of standard tableaux  $T$  such that  $T|_{[n]} = T_1$  and  $T|_{[n+1, n+m]} \approx T_2$ .

Given a tableau  $T$ , define  $\overline{T}$  to be the tableau of the same shape as  $T$  with reading word  $st(\mathbf{row}(T))$ . The following theorem is analogous to Theorem 1.1 and is not hard to prove using the methods of [10].

**Theorem 1.2.** *Let  $S$  be a standard Young tableau. We have*

$$\Delta(S) = \sum_{(T', T'') \in T(S)} \overline{T'} \otimes \overline{T''},$$

where  $T(S)$  is the set of pairs of tableaux  $T', T''$  such that  $\mathbf{row}(T')\mathbf{row}(T'') \approx \mathbf{row}(S)$ .

Let  $\Lambda$  denote the ring of symmetric functions. Denote by  $s_\lambda$  its basis of Schur functions, mentioned above. See for example [16] for details. Then  $\Lambda$  has a bialgebra structure, see [20] for details.

We are interested in a combinatorial rule for the coefficients  $c_{\lambda, \mu}^\nu$  in the decompositions

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda, \mu}^\nu s_\nu.$$

Define  $\psi : PR \longrightarrow \Lambda$  by

$$\psi(\mathbf{T}) = s_{\lambda(T)},$$

where  $\lambda(T)$  denotes the shape of  $T$ .

**Theorem 1.3.** [10, Theorem 5.4.5] *The map  $\psi$  is a bialgebra morphism.*

Applying  $\psi$  to the equalities in Theorem 1.1 and Theorem 1.2, we obtain the following two versions of the Littlewood-Richardson rule.

**Corollary 1.4.** [16, Theorem A1.3.1] *Let  $T$  be a standard Young tableau of shape  $\mu$ . Then the coefficient  $c_{\lambda, \mu}^\nu$  is equal to the number of standard Young tableaux  $R$  of skew shape  $\nu/\lambda$  such that  $\mathbf{row}(R) \approx \mathbf{row}(T)$ .*

**Corollary 1.5.** [10, Theorem 5.4.5] *Let  $S$  be a standard Young tableau of shape  $\nu$ . Then the coefficient  $c_{\lambda, \mu}^\nu$  in the decomposition is equal to the number of standard Young tableaux  $R$  of skew shape  $\lambda \oplus \mu$  such that  $\mathbf{row}(R) \approx S$ .*

**1.3.  $K$ -theoretic Poirier-Reutenauer bialgebra and Littlewood-Richardson rule.** The combinatorics of the  $K$ -theory of Grassmannians has been developed in [5, 6, 9]. In [1] Buch gave an explicit description of the *stable Grothendieck polynomials*, which represent Schubert classes in the  $K$ -theory ring. Such a description was already implicit in [5]. Then Buch proceeded to give a Littlewood-Richardson rule, which describes the structure constants of the ring with respect to the basis of those classes. An alternative description of those structure constants was obtained by Thomas and Yong in [18, 19].

In [2], a natural analogue of Knuth insertion called *Hecke insertion* is defined. A result of such insertion is an *increasing tableau*, which is a natural analogue of a standard Young tableau.

A question arises then: can one use Hecke insertion to define a  $K$ -theoretic analogue of the Poirier-Reutenauer Hopf algebra? Can one then proceed to obtain a version of the Littlewood-Richardson rule analogous to Corollary 1.4 and Corollary 1.5? It turns out the answer is yes, although there are additional obstacles to overcome. This is the goal of this paper.

It turns out that there is no *local* way to describe equivalence between words that Hecke insert into the same tableaux. This was, of course, already known in [2]. The consequence is

that the verbatim definition of the Poirier-Reutenauer bialgebra simply does not work. If for an increasing tableau  $T$  we define

$$\mathbf{T} = \sum_{P(w)=T} w,$$

where the sum is over all words that Hecke insert into  $T$ , the resulting sums are not closed under the natural product and coproduct, see Remark 3.8 and Remark 3.14.

We use instead classes defined by the *K-Knuth equivalence relation* of [3], a combination of the Hecke equivalence of [1] and Knuth equivalence. The relation is defined by the following three local rules:

$$\begin{aligned} pp &\equiv p && \text{for all } p \\ pq p &\equiv qp q && \text{for all } p \text{ and } q \\ pqs &\equiv qps \text{ and } sqp \equiv spq && \text{whenever } p < s < q. \end{aligned}$$

It is important to note that the *K-Knuth* classes combine some classes of increasing tableaux, as seen in [3]. In other words, there are *K-Knuth* equivalence classes of words that have more than one corresponding tableau. For example, the *K-Knuth* equivalence class of 3124 contains *two* increasing tableaux, shown below.

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}.$$

We invite the reader to verify that the row reading words of those tableaux can be indeed connected to each other by *K-Knuth* equivalences.

In order to get a working version of the Littlewood-Richardson rule, such tableaux need to be avoided. We use the notion of a *unique rectification target* of Buch and Samuel, see [3], which are increasing tableaux with the property of being the only increasing tableau in their *K-Knuth* equivalence class. We will refer to a unique rectification target as a URT.

Finally, armed with this notion of unique rectification targets, we can state and prove the following versions of the Littlewood-Richardson rule, similar to those of Corollary 1.4 and Corollary 1.5. The first was proven previously in [3, Corollary 3.19] and in less generality in [18, Theorem 1.2] using a K-theoretic analogue of jeu de taquin.

**Theorem** (Theorem 6.1). *Let  $T$  be a URT of shape  $\mu$ . Then the coefficient  $c_{\lambda,\mu}^\nu$  in the decomposition*

$$G_\lambda G_\mu = \sum_{\nu} (-1)^{|\nu| - |\lambda| - |\mu|} c_{\lambda,\mu}^\nu G_\nu$$

*is equal to the number of increasing tableaux  $R$  of skew shape  $\nu/\lambda$  such that  $P(\mathbf{row}(R)) = T$ .*

While we obtain the next result only for unique rectification targets, [19, Theorem 1.4] proves it for arbitrary increasing tableaux.

**Theorem** (Theorem 6.4). *Let  $T_0$  be a URT of shape  $\nu$ . Then the coefficient  $d_{\lambda,\mu}^\nu$  in the decomposition*

$$\Delta(G_\nu) = \sum_{\lambda,\mu} (-1)^{|\nu| - |\lambda| - |\mu|} d_{\lambda,\mu}^\nu G_\lambda \otimes G_\mu$$

*is equal to the number of increasing tableaux  $R$  of skew shape  $\lambda \oplus \mu$  such that  $P(\mathbf{row}(R)) = T_0$ .*

**Remark 1.6.** Let us make the relationship between the two previous theorems, the two theorems of Thomas and Yong: Theorem [19, Theorem 1.4] and [18, Theorem 1.4], and the result of Buch and Samuel [3, Corollary 3.19] clear. Their theorems are stated in terms of *K-theoretic* jeu

de taquin, which is introduced by Thomas and Yong. In [3], Buch and Samuel prove that  $K$ -Knuth equivalence is equivalent to  $K$ -theoretic jeu de taquin equivalence in [3, Theorem 6.2], thus explaining the connection.

Therefore, our Theorem 6.1 is a corollary of [19, Theorem 1.4], where our theorem is more specialized since we require  $T$  to be a URT. On the other hand, Theorem 6.4 is the same as [3, Corollary 3.10], which both generalize [18, Theorem 1.4], as we allow  $S$  to be an arbitrary URT rather than fixing a particular (*superstandard*) choice.

In our proof of an analogue of Theorem 1.3, it is more natural to work with the *weak set-valued tableaux* defined in [8] than with the set-valued tableaux of Buch [1]. However, as we show in Corollary 5.12, the two languages are equivalent.

**1.4. Plan of the paper and acknowledgements.** In Section 2, we describe Hecke insertion and reverse Hecke insertion. We define the insertion tableau,  $P(w)$ , and the recording tableau,  $Q(w)$ , for a word  $w$ . We review several relevant results regarding Hecke insertion. We then recall (from [3]) the  $K$ -Knuth equivalence of finite words on the alphabet  $\{1, 2, 3, \dots\}$  and discuss certain characteristics of this equivalence.

In Section 3, we define  $[[h]]$  to be the sum of all words in the Hecke equivalence class of a word  $h$ . We define a vector space,  $KPR$ , spanned by all such sums. We introduce a bialgebra structure on  $KPR$  and show that  $KPR$  has no antipode for this bialgebra structure. Thus, we obtain the  $K$ -theoretic Poirier-Reutenauer bialgebra.

In Section 4, we recall from [3] the notion of a *unique rectification target* (URT), a tableau that is the unique tableau in its  $K$ -Knuth equivalence class. We rephrase the product and coproduct formulas from the previous section for  $K$ -Knuth equivalence classes that correspond to URTs.

In Section 5, we draw a connection between the material in the previous sections and the ring of symmetric functions. We define the *stable Grothendieck polynomials*,  $G_\lambda$ , as in [1] by using set-valued tableaux and discuss their structure constants. We then use *weak set-valued tableaux* to define *weak stable Grothendieck polynomials*,  $J_\lambda$ . We show that the bialgebra structure constants of the  $G_\lambda$  and the  $J_\lambda$  coincide up to a sign. Using the fundamental quasisymmetric functions, we define a bialgebra morphism,  $\phi$ , with the property that  $\phi([[h]])$  can be written as a sum of weak stable Grothendieck polynomials.

In Section 6, we use the bialgebra morphism from Section 5 to state and prove a Littlewood-Richardson rule for the product and coproduct of the stable Grothendieck polynomials.

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## 2. HECKE INSERTION AND THE $K$ -KNUTH MONOID

**2.1. Hecke insertion.** An *increasing tableau* is a filling of a Young diagram with positive integers such that the entries in rows are strictly increasing from left to right and the entries in columns are strictly increasing from top to bottom.

**Example 2.1.** The tableau shown on the left is an increasing tableau. The tableau on the right is not an increasing tableau because the entries in the first row are not strictly increasing.

1	2	4	5
2	3	5	7
6	7		
8			

1	2	2	4
3	4		
5			

**Lemma 2.2.** *There are only finitely many increasing tableaux filled with a given finite alphabet.*

*Proof.* If the alphabet used has  $n$  letters, each row and each column cannot be longer than  $n$ .  $\square$

Of particular importance in what follows will be increasing tableaux on alphabets consisting of the first several positive integers, i.e. on  $[k] = \{1, 2, \dots, k\}$ . We call such increasing tableaux *initial*.

We follow [2] to give a description of Hecke (row) insertion of a positive integer  $x$  into an increasing tableau  $Y$  resulting in an increasing tableau  $Z$ . The shape of  $Z$  is obtained from the shape of  $Y$  by adding at most one box. If a box is added in position  $(i, j)$ , then we set  $c = (i, j)$ . In the case where no box is added, then  $c = (i, j)$ , where  $(i, j)$  is a special corner indicating where the insertion process terminated. We will use a parameter  $\alpha \in \{0, 1\}$  to keep track of whether or not a box is added to  $Y$  after inserting  $x$  by setting  $\alpha = 0$  if  $c \in Y$  and  $\alpha = 1$  if  $c \notin Y$ . We use the notation  $Z = (Y \xleftarrow{H} x)$  to denote the resulting tableau, and we denote the outcome of the insertion by  $(Z, c, \alpha)$ .

We now describe how to insert  $x$  into increasing tableau  $Y$  by describing how to insert  $x$  into  $R$ , a row of  $Y$ . This insertion may modify the row and may produce an output integer, which we will insert into the next row. To begin the insertion process, insert  $x$  into the first row of  $Y$ . The process stops when there is no output integer. The rules for insertion of  $x$  into  $R$  are as follows:

- (H1) If  $x$  is weakly larger than all integers in  $R$  and adjoining  $x$  to the end of row  $R$  results in an increasing tableau, then  $Z$  is the resulting tableau and  $c$  is the new corner where  $x$  was added.
- (H2) If  $x$  is weakly larger than all integers in  $R$  and adjoining  $x$  to the end of row  $R$  does not result in an increasing tableau, then  $Z = Y$ , and  $c$  is the box at the bottom of the column of  $Z$  containing the rightmost box of the row  $R$ .

For the next two rules, assume  $R$  contains boxes strictly larger than  $x$ , and let  $y$  be the smallest such box.

- (H3) If replacing  $y$  with  $x$  results in an increasing tableau, then replace  $y$  with  $x$ . In this case,  $y$  is the output integer to be inserted into the next row
- (H4) If replacing  $y$  with  $x$  does not result in an increasing tableau, then do not change row  $R$ . In this case,  $y$  is the output integer to be inserted into the next row.

**Example 2.3.**

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 & & & \\ \hline 7 & & & \\ \hline \end{array} \xleftarrow{H} 3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 & & & \\ \hline 7 & & & \\ \hline \end{array}$$

We use rule (H4) in the first row to obtain output integer 5. Notice that the 5 cannot replace the 6 in the second row since it would be directly below the 5 in the first row. Thus we use (H4) again and get output integer 6. Since we cannot add this 6 to the end of the third row, we use (H2) and get  $c = (1, 4)$ . Notice that the shape did not change in this insertion, so  $\alpha = 0$ .

**Example 2.4.**

$$\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 3 & 6 & 8 \\ \hline 7 & & \\ \hline \end{array} \xleftarrow{H} 5 = \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 3 & 6 & 8 \\ \hline 7 & 8 & \\ \hline \end{array}$$

The integer 5 bumps the 6 from the first row using (H3). The 6 is inserted into the second row, which already contains a 6. Using (H4), the second row remains unchanged and we insert 8 into the third row. Since 8 is larger than everything in the third row, we use (H1) to adjoin it to the end of the row. Thus  $\alpha = 1$ .

In [2], Buch, Kresch, Shimozono, Tamvakis, and Yong give the following algorithm for reverse Hecke insertion starting with the triple  $(Z, c, \alpha)$  as described above and ending with a pair  $(Y, x)$  consisting of an increasing tableau and a positive integer.

(rH1) If  $y$  is the cell in square  $c$  of  $Z$  and  $\alpha = 1$ , then remove  $y$  and reverse insert  $y$  into the row above.

(rH2) If  $\alpha = 0$ , do not remove  $y$ , but still reverse insert it into the row above.

In the row above, let  $x$  be the largest integer such that  $x < y$ .

(rH3) If replacing  $x$  with  $y$  results in an increasing tableau, then we replace  $x$  with  $y$  and reverse insert  $x$  into the row above.

(rH4) If replacing  $x$  with  $y$  does not result in an increasing tableau, leave the row unchanged and reverse insert  $x$  into the row above.

(rH5) If  $R$  is the first row of the tableau, the final output consists of  $x$  and the modified tableau.

**Theorem 2.5.** [2, Theorem 4] *Hecke insertion  $(Y, x) \mapsto (Z, c, \alpha)$  and reverse Hecke insertion  $(Z, c, \alpha) \mapsto (Y, x)$  define mutually inverse bijections between the set of pairs consisting of an increasing tableau and a positive integer and the set of triples consisting of an increasing tableau, a corner cell of the increasing tableau, and  $\alpha \in \{0, 1\}$ .*

Buch, Kresch, Shimozono, Tamvakis, and Yong prove the following lemma about Hecke insertion, which will be useful later.

**Lemma 2.6.** [2, Lemma 2] *Let  $Y$  be an increasing tableau and  $x_1, x_2$  be two positive integers. Suppose that Hecke insertion of  $x_1$  into  $Y$  results in  $(Z, c_1)$  and Hecke insertion of  $x_2$  into  $Z$  results in  $(T, c_2)$ . Then  $c_2$  is strictly below  $c_1$  if and only if  $x_1 > x_2$ .*

Define the *row reading word* of an increasing tableau  $T$ ,  $\text{row}(T)$ , to be its content read left to right in each row, starting from the bottom row and ending with the top row.

**Example 2.7.** The second tableau in Example 2.4 has the reading word 78368245.

Suppose  $w = w_1 w_2 \dots w_n$  is a word. Its *insertion tableau* is

$$P(w) = (\dots ((\emptyset \xleftarrow{H} w_1) \xleftarrow{H} w_2) \dots \xleftarrow{H} w_n).$$

We shall also need the following two lemmas.

**Lemma 2.8.** *If  $P(w) = T$  then  $P(w)|_{[k]} = P(w|_{[k]}) = T|_{[k]}$ .*

*Proof.* This follows from the insertion rules; letters greater than  $k$  never affect letters in  $[k]$ .  $\square$

**Lemma 2.9.** *For any tableau  $T$ ,  $P(\text{row}(T)) = T$ .*

*Proof.* It is easy to see that when each next row is inserted, it pushes down the previous rows.  $\square$

**2.2. Recording tableaux.** A *set-valued tableau*  $T$  of shape  $\lambda$  is a filling of the boxes with finite, non-empty subsets of positive integers so that

- (1) the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present), and
- (2) the smallest number in each box is strictly greater than the largest number in the box directly above it (if that box is present).

Given a word  $h = h_1 h_2 \dots h_l$ , we can associate a pair of tableaux  $(P(h), Q(h))$ , where  $P(h)$  is the insertion tableau described previously and  $Q(h)$  is a set-valued tableau called the *recording tableau* obtained as follows. Start with  $Q(\emptyset) = \emptyset$ . At each step of the insertion of  $h$ , let  $Q(h_1 \dots h_k)$  be obtained from  $Q(h_1 \dots h_{k-1})$  by labeling the special corner,  $c$ , in the insertion of  $h_k$  into  $P(h_1 \dots h_{k-1})$  with the positive integer  $k$ . Then  $Q(h) = Q(h_1 h_2 \dots h_l)$  is the resulting strictly increasing set-valued tableau.

**Example 2.10.** Let  $h$  be 15133. We obtain  $(P(h), Q(h))$  with the following sequence, where in column  $k$ ,  $Q(h_1 \dots h_k)$  is shown below  $P(h_1 \dots h_k)$ .

$$\begin{array}{ccccc}
\boxed{1} & \boxed{1 \ 5} & \boxed{1 \ 5} & \boxed{1 \ 3} & \boxed{1 \ 3} \\
& & \boxed{5} & \boxed{5} & \boxed{5}
\end{array} = P(h)$$

$$\begin{array}{ccccc}
\boxed{1} & \boxed{1 \ 2} & \boxed{1 \ 2} & \boxed{1 \ 2} & \boxed{1 \ 25} \\
& & \boxed{3} & \boxed{34} & \boxed{34}
\end{array} = Q(h)$$

Call a word  $h$  *initial* if the letters appearing in it are exactly the numbers in  $[k]$  for some positive integer  $k$ .

**Example 2.11.** The word 13422 is initial since the letters appearing in it are the numbers from 1 to 4. On the other hand, the word 1422 is not initial because the letters appearing in it are 1, 2, and 4 and do not form a set  $[k]$  for any  $k$ .

**Theorem 2.12.** *The map sending  $h = h_1 h_2 \dots h_n$  to  $(P(h), Q(h))$  is a bijection between words and ordered pairs of tableaux of the same shape  $(P, Q)$ , where  $P$  is an increasing tableau and  $Q$  is a set-valued tableau with entries  $\{1, 2, \dots, n\}$ . It is also a bijection if there is an extra condition of being initial imposed both on  $h$  and  $P$ .*

*Proof.* It is clear from the definition of  $Q(h)$  that  $P(h)$  and  $Q(h)$  have the same shape, and it is clear from the insertion algorithm that  $P(h)$  is an increasing tableau and  $Q(h)$  is an increasing set-valued tableau. Thus, we must show that given  $(P, Q)$ , one can uniquely recover  $h$ .

To recover  $h$ , perform reverse Hecke insertion in  $P$  multiple times as follows. Let  $l$  be the largest entry in  $Q$  and call its cell  $c(l)$ . If  $l$  is the only entry in  $c(l)$  inside in  $Q$ , perform reverse Hecke insertion with the triple  $(P, c(l), \alpha = 1)$ . If the  $l$  is not the only entry in its cell in  $Q$ , perform reverse Hecke insertion with the triple  $(P, c(l), \alpha = 0)$ . This reverse Hecke insertion will end with output  $(P_2, x_l)$ . Set  $Q_2 = Q - \{l\}$ , and follow the same procedure described above replacing  $Q$  with  $Q_2$  and  $P$  with  $P_2$ . The reverse insertion will end with output  $(P_3, x_{l-1})$ . Set  $Q_3 = Q_2 - \{l-1\}$ . Continue this process until the output tableau is empty. By Theorem 2.5,  $h = x_1 x_2 \dots x_l$ ,  $P(h) = P$ , and  $Q(h) = Q$ .  $\square$

**Example 2.13.** Let's start with the pair  $(P, Q)$  from the previous example and recover  $h$ .

$$P = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 5 & \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|} \hline 1 & 25 \\ \hline 34 & \\ \hline \end{array}$$

We first notice the largest entry of  $Q$  is in cell  $(1, 2)$  and is not the smallest entry in cell  $(1, 2)$ , so we perform the reverse Hecke insertion determined by the triple  $(P, (1, 2), 0)$ . The output of this reverse insertion is  $(P_2, 3)$ , so  $h_5 = 3$ .

$$P_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 5 & \\ \hline \end{array} \qquad Q_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 34 & \\ \hline \end{array}$$



The largest entry in  $Q_2$  is in cell  $(2, 1)$  and is not the smallest entry in cell  $(2, 1)$ , so we perform the reverse Hecke insertion determined by  $(P_2, (2, 1), 0)$  and obtain output  $(P_3, 3)$ . Thus  $h_4 = 3$ .

$$P_3 = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 5 & \\ \hline \end{array} \quad Q_3 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

The largest entry in  $Q_3$  is in cell  $(2, 1)$  and is the smallest entry in its cell. We perform reverse insertion  $(P_3, (2, 1), 1)$ , obtain output  $(P_4, 1)$ , and set  $h_3 = 1$ .

$$P_4 = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline & \\ \hline \end{array} \quad Q_4 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & \\ \hline \end{array}$$

In the last two steps, we recover  $h_2 = 5$  and  $h_1 = 1$ .

**2.3.  $K$ -Knuth equivalence.** We next introduce the  $K$ -Knuth monoid of [3] as the quotient of the free monoid of all finite words on the alphabet  $\{1, 2, 3, \dots\}$  by the following relations:

- (1)  $pp \equiv p$  for all  $p$
- (2)  $pqp \equiv qpq$  for all  $p$  and  $q$
- (3)  $pqs \equiv qps$  and  $sqp \equiv spq$  whenever  $p < s < q$ .

This monoid is better suited for our purposes than Hecke monoid of [2], see Remark 4.5.

We shall say two words are  $K$ -Knuth equivalent if they are equal in the  $K$ -Knuth monoid. We denote  $K$ -Knuth equivalence by  $\equiv$ . We shall also say two words are *insertion equivalent* if they Hecke insert into the same tableau. We shall denote insertion equivalence by  $\sim$ .

**Example 2.14.** The words 34124 and 3124 are  $K$ -Knuth equivalent, since

$$34124 \equiv 31424 \equiv 31242 \equiv 13242 \equiv 13422 \equiv 1342 \equiv 1324 \equiv 3124.$$

They are not insertion equivalent, however, since they insert into the following two distinct tableaux.

$$P(34124) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array} \quad P(3124) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}.$$

**Example 2.15.** As we soon shall see,  $13524 \not\equiv 15324$ .

**2.4. Properties of  $K$ -Knuth equivalence.** We will need three additional properties of Hecke insertion and  $K$ -Knuth equivalence. The first follows from [3, Theorem 6.2].

**Theorem 2.16.** *Insertion equivalence implies  $K$ -Knuth equivalence: if  $w_1 \sim w_2$  for words  $w_1$  and  $w_2$ , then  $w_1 \equiv w_2$ .*

As we saw in Example 2.14, the converse of this result is not true.

We now examine the length of the longest strictly increasing subsequence of a word  $w$ , denoted by  $\text{lis}(w)$ , and length of the longest strictly decreasing subsequence of  $w$ ,  $\text{lds}(w)$ . The next result follows from the  $K$ -Knuth equivalence relations.

**Lemma 2.17.** *If  $w_1 \equiv w_2$ , then  $\text{lis}(w_1) = \text{lis}(w_2)$  and  $\text{lds}(w_1) = \text{lds}(w_2)$ .*

*Proof.* It is enough to assume the two words differ by one equivalence relation.

Suppose  $w_1 = upv$  and  $w_2 = uppv$  for some possibly empty words  $u$  and  $v$ . Then if  $u'pv'$  is a strictly increasing sequence in  $w_1$ , for some possibly empty  $u'$  subword of  $u$  and  $v'$  subword of  $v$ , it is also a strictly increasing sequence in  $w_2$ . And since a strictly increasing sequence can only use one occurrence of  $p$ , any strictly increasing sequence in  $w_2$  is also strictly increasing in  $w_1$ .

Next, consider the case where  $w_1 = upqpv$  and  $w_2 = upqpq$ , and assume  $p < q$ . If  $u'pqv'$  is a strictly increasing sequence in  $w_1$ , notice that we have the same sequence in  $w_2$ . Similarly, if a

strictly increasing sequence of  $w_1$  is of the form  $u'pv'$  or  $u'qv'$ , we have the same sequence in  $w_2$ . Since strictly increasing sequences of  $w_2$  involving the  $p$  or  $q$  that are outside of  $u$  and  $v$  have the same form as those described above, any strictly increasing sequence of  $w_2$  appears as a strictly increasing sequence in  $w_1$ .

Lastly, suppose  $w_1 = upqsv$  and  $w_2 = uqpsv$  for  $p < s < q$ . If a strictly increasing sequence in  $w_1$  (resp.  $w_2$ ) uses only one of the  $p$  and  $q$  outside of  $u$  and  $v$ , then clearly this is still a strictly increasing sequence in  $w_2$  (resp.  $w_1$ ). If a strictly increasing sequence in  $w_1$  is  $u'pqv'$ , then  $u'psv'$  is a strictly increasing sequence in  $w_2$  of the same length and vice versa.

A similar argument applies for  $\mathbf{lds}(w_1)$  and  $\mathbf{lds}(w_2)$ .  $\square$

We can use this result to verify that 13524 is not  $K$ -Knuth equivalent to 15324, as promised in Example 2.15. Indeed,  $\mathbf{lds}(13524) = 2$  and  $\mathbf{lds}(15324) = 3$ .

**Remark 2.18.** We do not know any analogue of the other Greene-Kleitman invariants, see [7] and [17].

We shall need the following lemma.

**Lemma 2.19.** [3, Lemma 5.5] *Let  $I$  be an interval in the alphabet  $A$ . If  $w \equiv w'$ , then  $w|_I \equiv w'|_I$ .*

The last result in this section was proved by H. Thomas and A. Yong in [17]. It gives information about the shape of  $P(w)$  and of  $P(h)$  for any  $h \equiv w$ .

**Theorem 2.20.** [17, Theorem 1.3] *For any word  $w$ , the size of the first row and first column of its insertion tableau are given by  $\mathbf{lis}(w)$  and  $\mathbf{lds}(w)$ , respectively.*

### 3. $K$ -THEORETIC POIRIER-REUTENAUER

Let  $[[h]]$  denote the sum of all words in the  $K$ -Knuth equivalence class of an initial word  $h$ :

$$[[h]] = \sum_{h \equiv w} w.$$

This is an infinite sum. The number of terms in  $[[h]]$  of length  $l$  is finite, however, for every positive integer  $l$ .

Let  $KPR$  denote the vector space spanned by all sums of the form  $[[h]]$  for some initial word  $h$ . We will endow  $KPR$  with a product and a coproduct structure, which are compatible with each other. We will refer to the resulting bialgebra as the  *$K$ -theoretic Poirier-Reutenauer bialgebra* and denote it by  $KPR$ .

**3.1.  $K$ -Knuth equivalence of tableaux.** Suppose we have increasing tableaux  $T$  and  $T'$ . Recall that  $\mathbf{row}(T)$  denotes the row reading word of  $T$ . As in [3], we say that  $T \equiv T'$  if  $\mathbf{row}(T) \equiv \mathbf{row}(T')$ .

**Example 3.1.** For the  $T$  and  $T'$  shown below, we have that  $T \equiv T'$  because

$$\mathbf{row}(T) = 34124 \equiv \mathbf{row}(T') = 3124$$

as shown in Example 2.14.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array} \quad \equiv \quad T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}$$

Note that by Lemma 2.17 and Theorem 2.20, if two tableaux are equivalent, their first rows have the same size and their first columns have the same size.

The following lemma says that each element of  $KPR$  splits into insertion classes of words.

**Lemma 3.2.** *We have*

$$[[h]] = \sum_T \left( \sum_{P(w)=T} w \right)$$

where the sum is over all increasing tableaux  $T$  whose reading word is in the  $K$ -Knuth equivalence class of  $h$ .

*Proof.* Follows from Theorem 2.16. □

This expansion is always finite by Lemma 2.2.

**3.2. Product structure.** Let  $\sqcup\sqcup$  denote the usual shuffle product of words. Let  $h$  be a word in the alphabet  $[n]$ , and let  $h'$  be a word in the alphabet  $[m]$ . Denote by  $w[n]$  the word obtained from  $w$  by increasing each letter by  $n$ . Define

$$[[h]] \cdot [[h']] = \sum_{w \equiv h, w' \equiv h'} w \sqcup\sqcup w'[n].$$

**Theorem 3.3.** *For any two initial words  $h$  and  $h'$ , their product can be written as*

$$[[h]] \cdot [[h']] = \sum_{h''} [[h'']],$$

where the sum is over a certain set of initial words  $h''$ .

*Proof.* From Lemma 2.19, we know that if a word appears in the righthand sum, the entire equivalence class of this word appears as well. The claim follows. □

**Example 3.4.** Let  $h = 12$ ,  $h' = 312$ . Then

$$[[12]] \cdot [[312]] = [[53124]] + [[51234]] + [[35124]] + [[351234]] + [[53412]] + [[5351234]].$$

**Theorem 3.5.** *Let  $h$  be a word in alphabet  $[n]$ , and let  $h'$  be a word in alphabet  $[m]$ . Suppose  $\mathcal{T} = \{P(h), T'_1, T'_2, \dots, T'_s\}$  is the equivalence class containing  $P(h)$ . Then we have*

$$[[h]] \cdot [[h']] = \sum_{T \in T(h \sqcup\sqcup h')} \sum_{P(w)=T} w,$$

where  $T(h \sqcup\sqcup h')$  is the finite set of tableaux  $T$  such that  $T|_{[n]} \in \mathcal{T}$  and  $\text{row}(T)|_{[n+1, n+m]} \equiv h'[n]$ .

*Proof.* If  $w$  is a shuffle of some  $w_1 \equiv h$  and  $w_2 \equiv h'[n]$ , then by Lemma 2.8  $P(w)|_{[n]} = P(w_1) \in \mathcal{T}$ . By Lemma 3.2 and Theorem 3.3, we get the desired expansion. Its finiteness follows from Lemma 2.2. □

**Example 3.6.** Let's take  $h = 12$  and  $h' = 312$ . Then

$$P(h) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \text{ and } P(h') = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}.$$

The insertion tableaux appearing in their product are those shown below.

1	2	4
3		
5		

1	2	4
3	5	
5		

1	2	4
3	4	
5		

1	2	3	4
5			

1	2	4
3	5	

1	2	3	4
3	5		



Here,  $\emptyset$  should be understood to be the identity element of the ground field. We denote it by  $\emptyset$  so as to avoid confusion with the word 1.

**Theorem 3.10.** *For any initial word  $h$ , its coproduct can be written as*

$$\Delta([h]) = \sum_{h', h''} [[h']] \otimes [[h'']],$$

where the sum is over a certain set of pairs of initial words  $h', h''$ .

*Proof.* For  $w = a_1 a_2 \dots a_n$  let  $\blacktriangle(w) = \sum_{i=0}^n a_1 \dots a_i \otimes a_{i+1} \dots a_n$ , and  $\blacktriangle([h]) = \sum_{h \equiv w} \blacktriangle(w)$ . It is clear that  $\blacktriangle([h]) = \sum_{h', h''} [[h']] \otimes [[h'']]$  for some collection of pairs of words  $h', h''$ . This is because  $K$ -Knuth equivalence relations are local and thus can be applied on both sides of  $\otimes$  in parallel with applying the same relation to the corresponding word on the left. It remains to standardize every term on the right and to use the fact that  $K$ -Knuth equivalence relations commute with standardization.  $\square$

**Example 3.11.** If we take  $h = 12$  in the previous theorem, we have

$$\Delta([12]) = [\emptyset] \otimes [12] + [1] \otimes [1] + [12] \otimes [1] + [1] \otimes [12] + [12] \otimes [\emptyset].$$

**Theorem 3.12.** *Let  $h$  be a word. We have*

$$\Delta([h]) = \sum_{(T', T'') \in T(h)} \left( \sum_{P(w)=T'} w \right) \otimes \left( \sum_{P(w)=T''} w \right),$$

where  $T(h)$  is the finite set of pairs of tableaux  $T', T''$  such that  $\text{row}(T') \text{row}(T'') \equiv h$ .

*Proof.* As we have seen in the proof of Theorem 3.10,  $\blacktriangle([h]) = \sum_{h', h''} [[h']] \otimes [[h'']]$ . Note that the sum on the right is multiplicity-free and that if we split each of the  $[[h']]$  and  $[[h'']]$  into insertion classes, we get exactly

$$\sum_{(T', T'') \in T(h)} \left( \sum_{P(w)=T'} w \right) \otimes \left( \sum_{P(w)=T''} w \right).$$

It remains to apply standardization to get the desired result. The finiteness follows from Lemma 2.2.  $\square$

**Corollary 3.13.** *The vector space  $KPR$  is closed under the coproduct operation. That is, the sums appearing on the right hand side in Theorem 3.10 are always finite.*

*Proof.* Entries in the tableaux  $T$  and  $T'$  are a subset of letters in the word  $h$ . The statement follows from finiteness in Theorem 3.12 and the fact that  $K$ -Knuth classes are coarser than insertion classes.  $\square$

**Remark 3.14.** It is not true that insertion classes are closed under the coproduct. For example,  $123 \otimes 1$  is a term in  $\Delta(1342)$  and thus in the coproduct of its insertion class, but  $123 \otimes 11$  is not. To see this, consider all words  $h$  containing only 1, 2, 3, and 4 such that  $123 \otimes 11$  is in  $\Delta(h)$ . These words are 12344, 12433, 13422, and 23411, none of which are insertion equivalent to 1342.

**3.4. Compatibility and antipode.** Recall that a product  $\cdot$  and a coproduct  $\Delta$  are *compatible* if the coproduct is an algebra morphism:

$$\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y).$$

A vector space endowed with a compatible product and coproduct is called a *bialgebra*. We refer the reader for example to [15] for details on bialgebras.

**Theorem 3.15.** *The product and coproduct structures on  $KPR$  defined above are compatible, thus giving  $KPR$  a bialgebra structure.*

*Proof.* The result follows from the fact that the same is true for initial words. Indeed, if  $u = a_1 \dots a_n$  and  $w = b_1 \dots b_m$  are two initial words, then

$$\Delta(w \cdot u) = \Delta(w \sqcup u[n]) = \sum_{v=c_1 \dots c_{n+m}} \sum_{i=1}^{n+m} \overline{c_1 \dots c_i} \otimes \overline{c_{i+1} \dots c_{n+m}},$$

where  $v$  ranges over shuffles of  $w$  and  $u[n]$ . On the other hand,

$$\begin{aligned} \Delta(w) \cdot \Delta(u) &= \left( \sum_{i=1}^m \overline{b_1 \dots b_i} \otimes \overline{b_{i+1} \dots b_m} \right) \cdot \left( \sum_{j=1}^n \overline{a_1 \dots a_j} \otimes \overline{a_{j+1} \dots a_n} \right) = \\ &= \sum_{i,j} (\overline{a_1 \dots a_i} \sqcup \overline{b_1 \dots b_j[i]}) \otimes (\overline{a_{i+1} \dots a_n} \sqcup \overline{b_{j+1} \dots b_m[n-i]}). \end{aligned}$$

The two expressions are easily seen to be equal.  $\square$

We also remark that  $KPR$  has no antipode (see [15] for a definition). Indeed, assume  $S$  is an antipode. Then since

$$\Delta([1]) = \emptyset \otimes [1] + [1] \otimes \emptyset,$$

we solve

$$S([1]) = -\frac{[1]}{\emptyset + [1]}.$$

This final expression is not a finite linear combination of basis elements of  $KPR$ , and thus does not lie in the bialgebra.

#### 4. UNIQUE RECTIFICATION TARGETS

As we have seen,  $K$ -Knuth equivalence classes may have several corresponding insertion tableaux. The following is an open problem.

**Problem 4.1.** *Describe  $K$ -Knuth equivalence classes of increasing tableaux.*

Of special importance are the  $K$ -Knuth equivalence classes with only one element.

**4.1. Definition and examples.** We call  $T$  a *unique rectification target* or a URT if it is the only tableau in its  $K$ -Knuth equivalence class [3, Definition 3.5]. In other words,  $T$  is a URT if for every  $w \equiv \text{row}(T)$  we have  $P(w) = T$ . The terminology is natural in the context of the  $K$ -theoretic jeu de taquin of Thomas and Yong [17]. If  $P(w)$  is a URT, we call the equivalence class of  $w$  a unique rectification class.

For example,  $P(1342)$  is not a unique rectification target because  $3124 \equiv 34124$ , as shown in Example 2.14, and  $P(3124) \neq P(34124)$  as shown below.

$$P(3124) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \qquad P(34124) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array}$$

It follows that  $[3124]$  is not a unique rectification class.

In [3], Buch and Samuels give a uniform construction of unique rectification targets of any shape as follows. Define the *minimal increasing tableau*  $M_\lambda$  of shape  $\lambda$  by filling the boxes of  $\lambda$  with the smallest possible values allowed in an increasing tableau. In other words,  $M_\lambda$  is the tableau obtained by filling all boxes in the  $k$ th southwest to northeast diagonal of  $\lambda$  with positive integer  $k$ .

**Example 4.2.** The tableaux below are minimal increasing tableaux.

$$M_{(3,2,1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \hline 3 & & \\ \hline \end{array} \quad M_{(5,2,1,1)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 3 & & & \\ \hline 3 & & & & \\ \hline 4 & & & & \\ \hline \end{array}$$

In [18, Theorem 1.2], Thomas and Yong prove that the superstandard tableaux of shape  $\lambda$ ,  $S_\lambda$ , is a URT, where  $S_\lambda$  is defined to be the standard Young tableau with  $1, 2, \dots, \lambda_1$  in the first row,  $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2$  in the second row, etc.

**Example 4.3.** The following are superstandard tableaux.

$$S_{(3,2,1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array} \quad S_{(5,2,1,1)} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & & & \\ \hline 8 & & & & \\ \hline 9 & & & & \\ \hline \end{array}$$

**Problem 4.4.** Characterize all unique rectification targets or at least provide an efficient algorithm to determine if a given tableau is a URT.

**Remark 4.5.** Note that if one uses the less restrictive Hecke equivalence of [2] instead of the  $K$ -knuth equivalence of [3], URT are extremely scarce. For example, the tableau with reading word 3412 is equivalent to the tableau with reading word 3124. In fact, with this definition there is no standard URT of shape  $(2, 2)$ .

**4.2. Product and coproduct of unique rectification classes.** As we have seen before, the product and coproduct of insertion classes do not necessarily decompose into insertion classes. However, the story is different if the classes are unique rectification classes, as seen in the following theorems.

**Theorem 4.6.** Let  $T_1$  and  $T_2$  be two URT. Then we have

$$\left( \sum_{P(w)=T_1} w \right) \cdot \left( \sum_{P(w)=T_2} w \right) = \sum_{T \in T(T_1 \sqcup T_2)} \sum_{P(w)=T} w,$$

where  $T(T_1 \sqcup T_2)$  is the finite set of tableaux  $T$  such that  $T|_{[n]} = T_1$  and  $P(\text{row}(T))|_{[n+1, n+m]} = T_2$ .

*Proof.* Since  $T_1$  and  $T_2$  are URTs, the left hand side is  $([[\text{row}(T_1)]]) \cdot ([[\text{row}(T_2)]])$  and  $T(T_1 \sqcup T_2)$  is  $T(\text{row}(T_1) \sqcup \text{row}(T_2))$  as in Theorem 3.5.  $\square$

**Theorem 4.7.** Let  $T_0$  be a URT. We have

$$\Delta \left( \sum_{P(w)=T_0} w \right) = \sum_{(T', T'') \in T(T_0)} \left( \sum_{P(w)=\overline{T'}} w \right) \otimes \left( \sum_{P(w)=\overline{T''}} w \right),$$

where  $T(T_0)$  is the finite set of pairs of tableaux  $T', T''$  such that  $P(\text{row}(T') \text{row}(T'')) = T_0$ .

*Proof.* Since  $T_0$  is a URT, the left hand term is  $\Delta([[\text{row}(T_0)]])$  and  $T(T_0) = T(\text{row}(T_0))$  as described in Theorem 3.12.  $\square$

**Remark 4.8.** Note that a product of unique rectification classes is not necessarily a sum of unique rectification classes. For example, if we let  $w' = 12$  and  $w''[2] = 34$ , then 13422 appears in the shuffle product. One checks  $P(13422)$  is one of the tableaux in Example 2.14 and thus is not a URT.

Similarly, the coproduct of a unique rectification class does not necessarily decompose into unique rectification classes. Consider  $T_0$ ,  $T'$ ,  $T''$ , and  $T'''$  below.

$$T_0 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \quad T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline \end{array} \quad T'' = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array} \quad T''' = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 5 & \\ \hline \end{array}$$

One can check that  $T_0$  is a URT and  $P(\text{row}(T')\text{row}(T'')) = P(312524) = T_0$ , but  $T'$  is not a URT since it is equivalent to  $T'''$ .

## 5. CONNECTION TO SYMMETRIC FUNCTIONS

### 5.1. Symmetric functions and stable Grothendieck polynomials.

We denote the ring of symmetric functions in an infinite number of variables  $x_i$  by

$$\Lambda = \Lambda(x_1, x_2, \dots) = \bigoplus_{n \geq 0} \Lambda_n.$$

The  $n^{\text{th}}$  graded component,  $\Lambda_n$ , consists of homogenous elements of degree  $n$  with  $\Lambda_0 = \mathbb{R}$ . There are several important bases of  $\Lambda$  indexed by partitions  $\lambda$  of integers. The two most notable such bases are the monomial symmetric functions,  $m_\lambda$ , and Schur functions,  $s_\lambda$ . We refer the reader to [16] for definitions and further details on the ring  $\Lambda$ .

For  $f(x_1, x_2, \dots) \in \Lambda$  let

$$\Delta(f) = f(y_1, \dots; z_1, \dots) \in \Lambda(y_1, \dots) \otimes \Lambda(z_1, \dots)$$

be the result of splitting the alphabet of  $x_i$ 's into two disjoint alphabets of  $y_i$ 's and  $z_i$ 's.  $\Lambda$  is known to be a bialgebra with this coproduct, see [20].

Let us denote by  $\hat{\Lambda}$  the completion of  $\Lambda$ , which consists of possibly infinite linear combinations of  $m_\lambda$ 's. Each element of  $\hat{\Lambda}$  can be split into graded components, each being a finite linear combination of the  $m_\lambda$ 's. Also, let  $\hat{\Lambda} \hat{\otimes} \hat{\Lambda}$  denote the completion of  $\Lambda \otimes \Lambda$ , consisting of possibly infinite linear combinations of  $m_\lambda \otimes m_\mu$ 's. It is not hard to see that the completion  $\hat{\Lambda}$  inherits a bialgebra structure from  $\Lambda$  in the following sense.

**Theorem 5.1.** *If  $f, g \in \hat{\Lambda}$  then*

$$f \cdot g = \sum_{\mu} c_{\mu} m_{\mu} \in \hat{\Lambda}.$$

*If  $f \in \hat{\Lambda}$ , then*

$$\Delta(f) = \sum_{\mu, \nu} c_{\mu, \nu} m_{\mu} \otimes m_{\nu} \in \hat{\Lambda} \hat{\otimes} \hat{\Lambda}.$$

*Furthermore, the coefficients  $c_{\mu}$  and  $c_{\mu, \nu}$  in both expressions are unique.*

*Proof.* It is easy to see that each  $m_{\mu}$  in the first case and each  $m_{\mu} \otimes m_{\nu}$  in the second case can appear only in finitely many terms on the left. The claim follows.  $\square$

Recall the definition of a set-valued tableau given in Section 2.2. Given a set-valued tableau  $T$ , let  $x^T$  be the monomial in which the exponent of  $x_i$  is the number of occurrences of the letter  $i$  in  $T$ . Let  $|T|$  be the degree of this monomial.

**Example 5.2.** The tableau shown below has  $x^T = x_1 x_3 x_4 x_5^2 x_6^3 x_8 x_9$  and  $|T| = 11$ .



134	4	568
56	6	
9		

In [1], Buch proves a combinatorial interpretation of the single stable Grothendieck polynomials indexed by partitions,  $G_\lambda$ , which we present as the definition. This interpretation is implicitly present in the earlier paper [5].

**Theorem 5.3.** [1, Theorem 3.1] *The single stable Grothendieck polynomial  $G_\lambda$  is given by the formula*

$$G_\lambda = \sum_T (-1)^{|T| - |\lambda|} x^T,$$

where the sum is over all set-valued tableaux  $T$  of shape  $\lambda$ .

**Example 5.4.** We have

$$G_{(2,1)} = x_1^2 x_2 + 2x_1 x_2 x_3 - x_1^2 x_2^2 - 2x_1^2 x_2 x_3 - 8x_1 x_2 x_3 x_4 + \dots,$$

where, for example the coefficient of  $x_1^2 x_2 x_3$  is  $-2$  because of the tableaux shown below and the fact that for each of them,  $|T| - |\lambda| = 1$ .

1	12
3	

1	13
2	

The following claim is not surprising.

**Lemma 5.5.** *Each element  $f \in \hat{\Lambda}$  can uniquely be written as*

$$f = \sum_{\mu} c_{\mu} G_{\mu}.$$

Similarly, each element of  $\Lambda \hat{\otimes} \Lambda$  can uniquely be written as

$$\sum_{\mu, \nu} c_{\mu, \nu} G_{\mu} \otimes G_{\nu}.$$

*Proof.* Fix any complete order on monomials  $m_\lambda$  that agrees with the reverse dominance order for a fixed size  $|\lambda|$  and satisfies  $m_\mu < m_\lambda$  for  $|\mu| < |\lambda|$ . See, for example, [12] for details. Then  $m_\lambda$  is the minimal term of  $G_\lambda$ , and we can uniquely recover coefficients  $c_\mu$  by using  $G_\lambda$ 's to eliminate minimal terms in  $f$ . The proof of the second claim is similar.  $\square$

What is surprising, however, is the following two theorems proven by Buch in [1]. A priori, the products and the coproducts of  $G_\lambda$ 's do not have to decompose into *finite* linear combinations.

**Theorem 5.6.** [1, Corollary 5.5] *We have*

$$G_\lambda G_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} G_{\nu},$$

where the sum on the right is over a finite set of partition shapes  $\nu$ .

**Theorem 5.7.** [1, Corollary 6.7] *We have*

$$\Delta(G_\nu) = \sum_{\lambda, \mu} d_{\lambda, \mu}^{\nu} G_\lambda \otimes G_\mu,$$

where the sum on the right is over a finite set of pairs  $\lambda, \mu$ .

**5.2. Weak set-valued tableau.** A *weak set-valued tableau*  $T$  of shape  $\lambda$  is a filling of the boxes with finite, non-empty multisets of positive integers so that

- (1) the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present), and
- (2) the smallest number in each box is strictly greater than the largest number in the box directly above it (if that box is present).

Note that the numbers in each box are not necessarily distinct.

For a weak set-valued tableau  $T$ , define  $x^T$  to be  $\prod_{i \geq 1} x_i^{a_i}$ , where  $a_i$  is the number of occurrences of the letter  $i$  in  $T$ .

**Example 5.8.** The following weak set-valued tableau  $T$  has  $x^T = x_1^3 x_2^4 x_3^2 x_4^2 x_5 x_6 x_8$ .

11	12	2	346
223	45	8	

Let  $J_\lambda = \sum_T x^T$  denote the weight generating function of all weak set-valued tableaux  $T$  of shape  $\lambda$ . We will call  $J_\lambda$  the *weak stable Grothendieck polynomial* indexed by  $\lambda$ .

**Example 5.9.** We have that

$$J_{(2,1)} = x_1^2 x_2 + 2x_1 x_2 x_3 + 2x_1^3 x_2 + 3x_1^2 x_2^2 + 2x_1 x_2^3 + 8x_1 x_2 x_3 x_4 + \dots,$$

where, for example, the coefficient of  $x_1^2 x_2^2$  is 3 because of the following weak set-valued tableaux.

11	2
2	

1	12
2	

1	1
22	

**Remark 5.10.** In [8], weak stable Grothendieck polynomials  $J_\lambda$  were introduced when studying the effect of standard ring automorphism  $\omega$  on the stable Grothendieck polynomials  $G_\lambda$ . In particular, it was shown in [8, Theorem 9.21] that  $J_\lambda$  are symmetric functions. Note that our current convention for labeling  $J_\lambda$  differs from that in [8] by shape transposition.

**Theorem 5.11.** *We have*

$$J_\lambda(x_1, x_2, \dots) = (-1)^{|\lambda|} G_\lambda \left( \frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \dots \right).$$

*Proof.* There is a correspondence between set-valued tableaux and weak set-valued tableaux as follows. For each set-valued tableau  $T$ , we can obtain a family of weak set-valued tableaux of the same shape, call the family  $\mathcal{T}$ , by saying that  $W \in \mathcal{T}$  if and only if  $W$  can be constructed from  $T$  by turning subsets in boxes of  $T$  into multisets. Conversely, given any weak set-valued tableau, we can find the set-valued tableau it corresponds to by transforming its multisets into subsets containing the same positive integers. For example, if we have the  $T$  shown below, then  $W_1$  and  $W_2$  are in  $\mathcal{T}$ .

$T =$	<table><tr><td>13</td><td>4</td><td>57</td></tr><tr><td>4</td><td>6</td><td></td></tr></table>	13	4	57	4	6		$W_1 =$	<table><tr><td>133</td><td>4</td><td>57</td></tr><tr><td>4</td><td>666</td><td></td></tr></table>	133	4	57	4	666		$W_2 =$	<table><tr><td>13</td><td>4</td><td>557</td></tr><tr><td>44</td><td>66</td><td></td></tr></table>	13	4	557	44	66	
13	4	57																					
4	6																						
133	4	57																					
4	666																						
13	4	557																					
44	66																						

Thus if  $x^T = x_1^{a_1} x_2^{a_2} x_3^{a_3} \dots$ , we have

$$\sum_{W \in \mathcal{T}} x^W = \left( \frac{x_1}{1-x_1} \right)^{a_1} \left( \frac{x_2}{1-x_2} \right)^{a_2} \left( \frac{x_3}{1-x_3} \right)^{a_3} \dots$$

since we can choose to repeat any of the  $a_i$   $i$ 's any positive number of times.

Therefore

$$(-1)^{|\lambda|} G_\lambda \left( \frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \dots \right) = (-1)^{|\lambda|} \sum_T (-1)^{|T|-|\lambda|} \left( \frac{-x}{1-x} \right)^T = \sum_T \left( \frac{x}{1-x} \right)^T = J_\lambda(x_1, x_2, \dots),$$

where the sum is over set-valued tableaux  $T$

□

**Corollary 5.12.** *The structure constants of the rings with bases  $G_\lambda$  and  $J_\lambda$  coincide up to sign. In other words,*

$$J_\lambda J_\mu = \sum_\nu c_{\lambda,\mu}^\nu J_\nu$$

if and only if

$$G_\lambda G_\mu = \sum_\nu (-1)^{|\nu|-|\lambda|-|\mu|} c_{\lambda,\mu}^\nu G_\nu.$$

*Proof.* In one direction it is clear, in the other follows from

$$G_\lambda(x_1, x_2, \dots) = (-1)^{|\nu|-|\lambda|-|\mu|} J_\lambda \left( \frac{-x_1}{1-x_1}, \frac{-x_2}{1-x_2}, \dots \right).$$

□

**5.3. Fundamental quasisymmetric functions.** A composition of  $n$  is an ordered arrangement of positive integers which sum to  $n$ . For example,  $(3)$ ,  $(1, 2)$ ,  $(2, 1)$ , and  $(1, 1, 1)$  are all of the compositions of 3. To a composition  $\alpha$  of  $n$ , we associate  $S_\alpha \subset [n-1]$  by letting

$$S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{k-1}\}.$$

Conversely, if  $S = \{s_1, s_2, \dots, s_k\}$  is a subset of  $[n-1]$ , we associate a composition,  $\mathcal{C}(S)$ , to  $S$  by

$$\mathcal{C}(S) = \{s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_k\}.$$

For example, if  $S = (1, 4, 5) \subset [6-1]$ ,  $\mathcal{C}(S) = (1, 4-1, 5-4, 6-5) = (1, 3, 1, 1)$ . Conversely, given composition  $\alpha = (1, 3, 1, 1)$ ,  $S_\alpha = (1, 1+3, 1+3+1) = (1, 4, 5)$ .

We define the *descent set* of a word  $h = h_1 h_2 \dots h_l$  to be the set  $\mathcal{D}(H) = \{i | h_i > h_{i+1}\}$ . Then, using the definitions above, given that  $\mathcal{D}(h) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ , we have the associated composition

$$\mathcal{C}(h) = \mathcal{C}(\mathcal{D}(h)) = (\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_m - \alpha_{m-1}, l - \alpha_m).$$

We call  $\mathcal{C}(h)$  the *descent composition* for  $h$ .

**Example 5.13.** If  $h = 11423532$ , the descent set of  $h$  is  $\{3, 6, 7\}$  and  $\mathcal{C}(h) = (3, 3, 1, 1)$ .

We shall now define the fundamental quasisymmetric function,  $L_\alpha$ . Given  $\alpha$ , a composition of  $n$ , define

$$L_\alpha = \sum_{\substack{i_1 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S_\alpha}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

For more information on the ring of quasisymmetric functions and on fundamental quasisymmetric functions see [16].

**Example 5.14.** The fundamental quasisymmetric function indexed by the composition  $(1, 3)$  is

$$L_{(1,3)} = x_1x_2^3 + x_1x_3^3 + x_2x_3^3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1x_2x_3x_4 + \dots,$$

an infinite sum where all terms have degree 4. Note that every term must have  $i_1 < i_2$  since  $S_{(1,3)} = \{2\}$ , so  $x_1^2x_2^2$  will never appear in  $L_{(1,3)}$ .

Given a weak set-valued tableau  $T$  filled with elements of  $[n]$ , each appearing once, we say that there is a descent at entry  $i$  if  $i + 1$  is strictly below  $i$ . We may then find the descent set of  $T$  and determine the composition corresponding to its descent set,  $\mathcal{C}(T)$ , by listing the entries in increasing order and marking the entries at which there was a descent in the tableau.

**Example 5.15.** The descent set of  $T$  shown below is  $\{3, 5, 9\}$  and the corresponding composition is  $\mathcal{C}(T) = (3, 2, 4, 2)$ .

123	5	89	11
46	7	10	

Given any weak set-valued tableau  $T$ , we determine  $\mathcal{C}(T)$  by first standardizing tableau  $T$ . To standardize  $T$ , first find all  $c_1$  occurrences of 1 in  $T$  and replace them from southwest to northeast with  $1, 2, \dots, c_1$ . Next, replace the  $c_2$  2's from southwest to northeast with  $c_1 + 1, c_1 + 2, \dots, c_1 + c_2$ . Continue this process, replacing the  $c_i$   $i$ 's from southwest to northeast with the next available consecutive integer. The resulting tableau is the standardization of  $T$ . We may then find the descent set of the standardization and let  $\mathcal{C}(T)$  be the associated composition.

**Example 5.16.** The standardization of the weak set-valued tableau shown below is the tableau  $T'$  of Example 5.15. Thus  $\mathcal{C}(T') = \mathcal{C}(T) = (3, 2, 4, 2)$ .

$$T' = \begin{array}{|c|c|c|c|} \hline 122 & 3 & 56 & 8 \\ \hline 34 & 4 & 7 & \\ \hline \end{array}$$

Recall from Section 2.2 that  $Q(h)$  denotes the recording tableau of Hecke insertion.

**Theorem 5.17.** *Let  $h$  be a word and  $Q(h)$  be its recording tableau. Then  $\mathcal{C}(h) = \mathcal{C}(Q(h))$ .*

*Proof.* According to Lemma 2.6, there is a descent at position  $i$  of word  $h$  if and only if the entry  $i + 1$  is strictly below entry  $i$  in  $Q(h)$ .  $\square$

**Example 5.18.** Consider  $h = 13324535$  with  $P(h)$  and  $Q(h)$  shown below.

$P(h) =$	<table><tr><td>1</td><td>2</td><td>3</td><td>5</td></tr><tr><td>3</td><td>4</td><td></td><td></td></tr></table>	1	2	3	5	3	4			$Q(h) =$	<table><tr><td>1</td><td>23</td><td>5</td><td>68</td></tr><tr><td>4</td><td>7</td><td></td><td></td></tr></table>	1	23	5	68	4	7		
1	2	3	5																
3	4																		
1	23	5	68																
4	7																		

One easily checks that  $\mathcal{C}(h) = (3, 3, 2) = \mathcal{C}(Q(h))$ .

#### 5.4. Decomposition into fundamental quasisymmetric functions.

**Theorem 5.19.** *For any fixed increasing tableau  $T$  of shape  $\lambda$  we have*

$$J_\lambda = \sum_{P(h)=T} L_{\mathcal{C}(h)}.$$

*Proof.* We give an explicit weight-preserving bijection between the set of weak set-valued tableaux of shape  $\lambda$  and the set of pairs  $(h, \sigma')$  where  $h = h_1h_2 \dots h_l$  is a word with  $P(h) = T$  and  $\sigma'$  is a sequence of positive integers  $(i_1 \leq i_2 \leq \dots \leq i_l)$ , where  $i_j < i_{j+1}$  if  $j \in \mathcal{D}(h)$ .

Suppose we have a weak set-valued tableau  $W$  of shape  $\lambda$ . To obtain  $h$ , first standardize  $W$ . Next, using  $T$  as  $P(h)$  and the standardization of  $W$  as  $Q(h)$ , perform reverse Hecke insertion.

Let the entries of  $W$  in increasing order be  $i_1, i_2, i_3, \dots, i_l$ , where each  $i_j$  is a positive integer, and denote  $\sigma' = (i_1, i_2, \dots, i_l)$ . We then have  $i_j \leq i_{j+1}$  for all  $j \in \{1, 2, \dots, l-1\}$ , and  $i_j < i_{j+1}$  if  $j \in \mathcal{D}(h)$  by Theorem 5.17.

For the reverse map, suppose we have  $h = h_1 h_2 \dots h_l$  with  $P(h) = T$  and some

$$\sigma' = (i_1 \leq i_2 \leq \dots \leq i_l).$$

Then let  $W$  be the recording tableau of the insertion of  $h$ , which uses the positive integers of  $\sigma'$ , i.e.  $i_j$  is used to label the special corner  $c$  of the insertion  $P(h_1 h_2 \dots h_{j-1}) \xleftarrow{H} h_j$ . According to Lemma 2.6, the result is a valid weak set-valued tableau. Using Theorem 2.12 we conclude we indeed have a bijection.

It remains to note that for a fixed  $h$ , we have

$$\sum_{\sigma'=(i_1 \leq i_2 \leq \dots \leq i_l)} \prod_{j=1}^l x_{i_j} = L_{\mathcal{C}(h)},$$

where the sum is over  $\sigma'$  such that  $i_j < i_{j+1}$  if  $j \in \mathcal{D}(h)$ . □

**Example 5.20.** Suppose we start with increasing tableau  $T$  and the weak set-valued tableau  $W$  shown below.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \qquad W = \begin{array}{|c|c|c|c|} \hline 122 & 235 & 58 & \\ \hline 45 & 677 & & \\ \hline \end{array}$$

Performing reverse Hecke insertion with the standardization of  $W$  recording the special box  $c$  at each step, we obtain

$$h = 1114412252233.$$

The corresponding composition is  $(5, 4, 4)$ , and

$$\sigma' = (1, 2, 2, 2, 3, 4, 5, 5, 5, 6, 7, 7, 8).$$

To understand the inverse map, simply let  $T = P(h)$  and record the special box  $c$  at each step with the positive integers of  $\sigma'$  to obtain the corresponding weak set-valued tableau.

**Remark 5.21.** Pairs  $(h, \sigma')$  as above are an analogue of *biwords* of [10].

**Remark 5.22.** The decomposition of weak stable Grothendieck polynomials  $J_\lambda$  into fundamental quasisymmetric functions is similar to Stanley's theory of  $P$ -partitions, see [16]. A different  $K$ -theoretic analog of such decomposition appears in [8].

**5.5. Map from the  $KPR$  to symmetric functions.** Consider the map  $\phi : KPR \rightarrow Sym$  given by

$$[[h]] \mapsto \sum_{w \equiv h} L_{\mathcal{C}(w)}.$$

**Theorem 5.23.** *Map  $\phi$  is a bialgebra morphism.*

*Proof.* First we show the map preserves the product. Note that

$$L_{\mathcal{C}(w')} \cdot L_{\mathcal{C}(w'')} = \sum_{w \in \text{Sh}(w', w''[n])} L_{\mathcal{C}(w)},$$

21

where the sum is over all shuffles of  $w'$  and  $w''[n]$  (see [8]). Thus,

$$\begin{aligned}
\phi([h_1] \cdot [h_2]) &= \phi\left(\sum_{w' \equiv h_1, w'' \equiv h_2} w' \sqcup w''\right) \\
&= \sum_{w' \equiv h_1, w'' \equiv h_2} L_{\mathcal{C}(w' \sqcup w'')} \\
&= \left(\sum_{w' \equiv h_1} L_{\mathcal{C}(w')}\right) \left(\sum_{w'' \equiv h_2} L_{\mathcal{C}(w'')}\right) = \phi([h_1])\phi([h_2]).
\end{aligned}$$

For the coproduct, we show the result for  $\phi$  applied to  $w \equiv h$  with  $\phi(w) = L_{\mathcal{C}(w)}$  using that  $\Delta(L_{\mathcal{C}(w)}) = \sum L_\beta \otimes L_\gamma$ , where we sum over all  $\beta = (\beta_1, \dots, \beta_k)$  and  $\gamma = (\gamma_1, \dots, \gamma_n)$  such that  $(\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_n) = \mathcal{C}(w)$  or  $(\beta_1, \dots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \dots, \gamma_n) = \mathcal{C}(w)$  (see [8]). Then, for any  $w \equiv h$ ,

$$\Delta(\phi(w)) = \Delta(L_{\mathcal{C}(w)}) = \sum L_\beta \otimes L_\gamma = \phi \otimes \phi(\Delta(w))$$

because the terms in  $\Delta(w) = \sum_{i=0}^{|w|} w_1 \cdots w_i \otimes w_{i+1} \cdots w_{|w|}$  with  $i \in \mathcal{D}(w)$  give exactly the terms  $L_\beta \otimes L_\gamma$  where  $(\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_n) = \mathcal{C}(w)$  and all terms with  $i \notin \mathcal{D}(w)$  give exactly the terms  $L_\beta \otimes L_\alpha$  where  $(\beta_1, \dots, \beta_{k-1}, \beta_k + \gamma_1, \gamma_2, \dots, \gamma_n) = \mathcal{C}(w)$ . Thus  $\Delta(\phi([h])) = \phi \otimes \phi(\Delta([h]))$ .  $\square$

**Theorem 5.24.** *We have*

$$\phi([h]) = \sum_{\text{row}(T) \equiv h} J_{\lambda(T)},$$

where the sum is over all tableaux  $K$ -Knuth equivalent to  $[h]$ , and  $\lambda(T)$  denotes the shape of  $T$ .

*Proof.* Using Theorem 5.19, we have

$$\phi([h]) = \sum_{w \equiv h} L_{\mathcal{C}(w)} = \sum_{T \equiv P(h)} \sum_{P(w)=T} L_{\mathcal{C}(w)} = \sum_{T \equiv P(h)} J_{\lambda(T)} = \sum_{\text{row}(T) \equiv h} J_{\lambda(T)}.$$

$\square$

## 6. LITTLEWOOD-RICHARDSON RULE

### 6.1. LR rule for Grothendieck polynomials.

**Theorem 6.1.** *Let  $T$  be a URT of shape  $\mu$ . Then the coefficient  $c_{\lambda, \mu}^\nu$  in the decomposition*

$$G_\lambda G_\mu = \sum_{\nu} (-1)^{|\nu| - |\lambda| - |\mu|} c_{\lambda, \mu}^\nu G_\nu$$

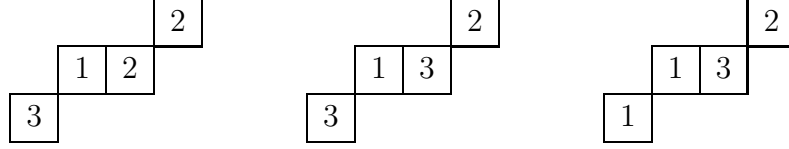
*is equal to the number of increasing tableaux  $R$  of skew shape  $\nu/\lambda$  such that  $P(\text{row}(R)) = T$ .*

*Proof.* In addition to the URT  $T$  of shape  $\mu$ , fix a URT  $T'$  of shape  $\lambda$ . Then by Theorems 4.6, 5.24, and 5.23,

$$\begin{aligned}
J_\lambda J_\mu &= \phi([\text{row}(T')])\phi([\text{row}(T)]) \\
&= \phi([\text{row}(T')] \cdot [\text{row}(T)]) \\
&= \phi\left(\sum_{Y \in T(T' \sqcup T)} \sum_{P(w)=Y} w\right) \\
&= \sum_{Y \in T(T' \sqcup T)} J_{\lambda(Y)},
\end{aligned}$$

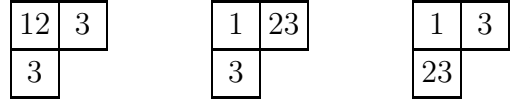
where  $T(T' \sqcup T)$  is the finite set of tableaux  $Y$  such that  $T|_{[\lambda]} = T'$  and  $P(\mathbf{row}(Y))|_{[\lambda+1, |\lambda|+|\mu|]} = T$ . Thus the coefficient of  $J_\nu$  in the product is the number of increasing tableaux  $R$  of skew shape  $\nu/\lambda$  such that  $P(\mathbf{row}(R)) = T$ . The desired result follows from Corollary 5.12.  $\square$

**Example 6.2.** The coefficient of  $G_{(4,3,1)}$  in  $G_{(3,1)}G_{(2,1)}$  is  $-3$ . To see this, fix  $T$  to be the tableau with reading word 312 as in the previous example, note  $(-1)^{|\nu|-|\lambda|-|\mu|} = -1$ , and notice the tableaux shown below are the only tableaux of shape  $(4,3,1)/(3,1)$  with  $P(\mathbf{row}(R)) = T$ .



Note that the claim may be false if  $T$  is not a URT.

**Example 6.3.** Suppose we want to find the coefficient of  $(4,2)$  in the product of  $(4,3,2)$  and  $(2,1)$ . Using Buch's rule [1], we compute that the coefficient is 3, corresponding to the following fillings of  $(2,1)$ :



However, if we choose the filling of  $(3,2)$  with row reading word 34124, one can easily check that there are only two ways to fill  $(4,3,2)/(2,1)$  with words equivalent to 34124 that insert into the chosen filling of  $(3,2)$ . The fillings are shown below.

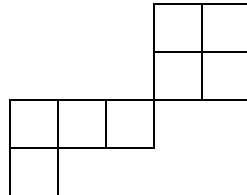


Now we can give our own proof of Theorem 5.6.

*Proof.* Combine Theorem 5.23 with Corollary 3.7.

Alternatively, the argument can be made directly from Theorem 6.1. Note that the set of shapes  $\nu$  such that there exists an increasing tableau  $R$  of skew shape  $\nu/\lambda$  such that  $P(\mathbf{row}(R)) = T$  is finite. This is because each cell in  $\nu/\lambda$  can be filled only with letters occurring in  $T$ , and thus size of each row and column in  $\nu/\lambda$  is bounded.  $\square$

**6.2. Dual LR rule for Grothendieck polynomials.** Given two Young diagrams,  $\lambda$  and  $\mu$ , define skew shape  $\lambda \oplus \mu$  to be the skew shape obtained by putting  $\lambda$  and  $\mu$  corner to corner. For example, The figure below shows  $(3,1) \oplus (2,2)$ .



**Theorem 6.4.** Let  $T_0$  be a URT of shape  $\nu$ . Then the coefficient  $d_{\lambda,\mu}^\nu$  in the decomposition

$$\Delta(G_\nu) = \sum_{\lambda,\mu} (-1)^{|\nu|-|\lambda|-|\mu|} d_{\lambda,\mu}^\nu G_\lambda \otimes G_\mu$$

is equal to the number of increasing tableaux  $R$  of skew shape  $\lambda \oplus \mu$  such that  $P(\mathbf{row}(R)) = T_0$ .

*Proof.* We have that

$$\begin{aligned}
\Delta(J_\nu) &= \Delta(\phi([[{\bf row}(T_0)]])) \\
&= \phi \otimes \phi(\Delta([[{\bf row}(T_0)]])) \\
&= \phi \otimes \phi \left( \sum_{(T', T'') \in T(T_0)} \sum_{P(w) = \overline{T'}} w \otimes \sum_{P(w) = \overline{T''}} w \right) \\
&= \sum_{(T', T'') \in T(T_0)} \phi([[{\bf row}(\overline{T'})]]) \otimes \phi([[{\bf row}(\overline{T''})]]) \\
&= \sum_{(T', T'') \in T(T_0)} J_{\lambda(T')} \otimes J_{\lambda(T'')},
\end{aligned}$$

where  $T(T_0)$  is the finite set of pairs of tableaux  $T', T''$  such that  $P(\text{row}(T')\text{row}(T'')) = T_0$ . Letting  $R = T' \oplus T''$ , the coefficient of  $J_\lambda \otimes J_\mu$  is exactly the number of increasing tableaux  $R$  of skew shape  $\lambda \oplus \mu$  such that  $P(\text{row}(R)) = T_0$ . The desired result follows from Corollary 5.12.  $\square$

**Example 6.5.** Fix  $T_0$  to be the URT of shape  $(3, 2)$  with reading word 45123. The coefficient of  $G_{(2,1)} \otimes G_{(2,1)}$  in  $G_{(3,2)}$  is  $-3$  because of the following three tableaux of shape  $(2, 1) \oplus (2, 1)$ .

Note that the claim may be false if  $T_0$  is not a URT.

**Example 6.6.** Suppose we have

$$T_0 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \\ \hline \end{array}.$$

We saw in Example 2.14 that  $T_0$  is not a URT. Now let  $\lambda = (2, 1)$  and  $\mu = (3, 1)$ . According to Buch's rule in [1], the coefficient of  $G_\lambda \otimes G_\mu$  in  $\Delta(G_{(3,2)})$  is at least 1 due to the following set-valued tableau:

1	1	3
23	34	

However, one can check that there is no skew tableau  $R$  of shape  $(2, 1) \oplus (3, 1)$  with  $P(\mathbf{row}(R)) = T_0$ .

Now we can give our own proof of Theorem 5.7.

*Proof.* Combine Theorem 5.23 with Corollary 3.13.

Alternatively, the argument can be made directly from Theorem 6.4. Note that the number of pairs  $\lambda, \mu$  such that there exists an increasing tableaux  $R$  of skew shape  $\lambda \oplus \mu$  such that  $P(\text{row}(R)) = T_0$  is finite. This is because each  $\lambda$  and  $\mu$  has to be filled with alphabet of  $T_0$  only, hence we can apply Lemma 2.2.  $\square$



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